Teaching towards Big Ideas: A review from the horizon

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To understand what teachers need to teach towards big ideas in the classroom, there is a need to systematically interface different conceptions of big ideas in mathematics with models of teacher knowledge. We conducted a literature review on horizon knowledge and big ideas to clarify both constructs and their relationships. Twenty-one journal articles were initially shortlisted, with within-case and cross-case analysis finally performed on four articles after inclusion/exclusion criteria. While it is clear that more work needs to be done, we tentatively conclude that to teach towards big ideas is to emphasise disciplinary ways of thinking that are empirically demonstrable to be fruitful for the learning of mathematics.

Teaching towards big ideas is a key shift in Singapore's most recent mathematics curriculum revision, implemented in 2020 (Toh et al., 2019; Choy, 2019). Teaching towards big ideas may present a huge pedagogical challenge for teachers. Firstly, there is a lack of clarity about what big ideas are. Although Charles (2005) defines a big idea as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10), different conceptions of big ideas continue to abound both in literature and in the practice of teaching. For example, the big idea of equivalence has a 'bigness' that can range from an understanding of the equal sign to the logical equivalence underlying every step in a series of algebraic manipulations to the equivalence relations that appear much beyond the domain of mathematics. Secondly, it is not clear what is meant by teaching towards big ideas. Some researchers highlight the importance of making explicit both "big content ideas" and "big process ideas" during lessons (Hurst 2015a). Others highlight the importance of reflecting on "issues of student learning and engagement as well as the domain", allowing mathematically worthwhile learning experiences to emerge from the connection of numerous smaller ideas (Mitchell et al., 2017). Such conceptions of big ideas may even seem not too different from existing understandings of expert teaching (Choy, 2019).

In a crowded curriculum, teachers may be tempted to force-fit the teaching of big ideas directly rather than teaching towards big ideas. How teachers can understand and appropriate the new notion of teaching towards big ideas, and yet, maintain the coherence and connection with their current pedagogical practices will depend on their mathematical knowledge for teaching (Ball et al., 2008). Ball et al. (2008)'s conceptions about Mathematical Knowledge for Teaching (MKT) make a distinction between Pedagogical Content Knowledge (PCK) and Subject Matter Knowledge (SMK). In particular, the notion of Horizon Content Knowledge (HCK), a component of SMK, resonates with 'teaching towards big ideas' since it isolates those aspects of mathematical knowledge which constitute an "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). The idea of seeing connections and coherence within and between mathematical topics may provide a way for teachers to navigate the challenges of teaching towards big ideas. However, like the notion of big ideas, 'horizon knowledge' has been defined differently and utilised in various ways (e.g., Jakobsen et al., 2014). Teaching towards big ideas requires teachers to present mathematics as a "coherent and connected enterprise" (NCTM, 2000, p. 17). To do this, there is a need to have some clarity regarding

the kind of knowledge needed. This begs the following research question: *How does the construct of 'horizon content knowledge' explicate how teachers can teach towards big ideas in the mathematics classroom?*

To answer this question, we adopted a systematic approach towards reviewing the literature discussing both 'horizon knowledge' and big ideas in conjunction. In particular, this research question is framed by addressing how the constructs of 'big ideas in mathematics' and 'horizon content knowledge' are respectively conceptualised in the mathematics education literature.

Method

Taking into account the best practices for conducting a systematic review (Alexander 2020; Siddaway et al. 2019, 2019), we took a systematic approach towards conducting a literature review by searching through four databases: EBSCO's Academic Search Complete, British Education Index, Education Source, and ERIC. Using a search term for 'big idea' or big ideas in "All Text", a total of 929 articles were initially obtained. A further refinement for texts that also contain 'horizon knowledge', 'horizon content knowledge', or 'mathematical horizon' yielded a total of 21 articles using Boolean search. We applied our inclusion/exclusion criteria to obtain four articles for our focus, as summarised in the chart (see Figure 1).
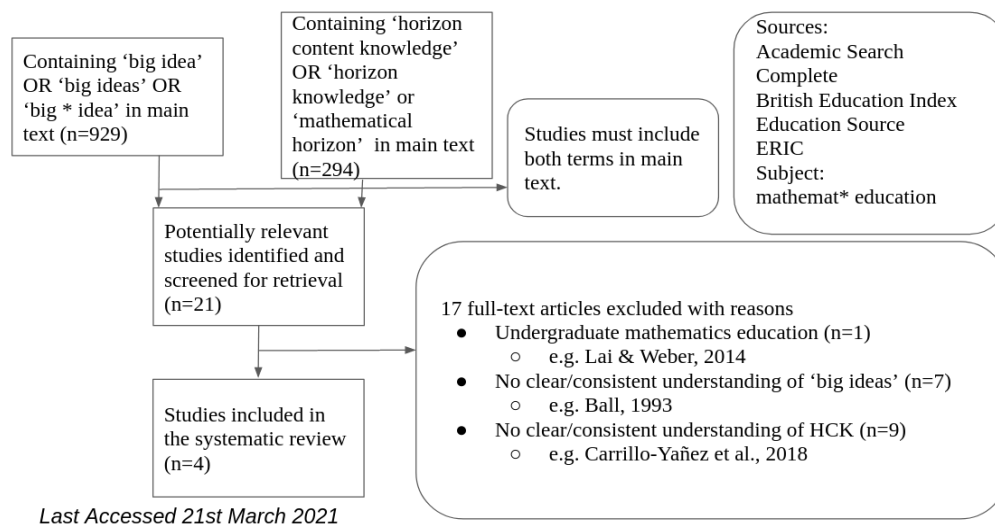

Figure 1. Systematic inclusion and exclusion

An example of a journal article with no clear understanding of HCK offered is Carrilo-Yanez et al. (2018). The article proposes a new model of mathematical knowledge. On one hand, the Mathematics Teacher's Specialised Knowledge (MTSK) has a component Knowledge of the structure of mathematics (KSM) that could be coded for big ideas with its distinction between 'intra-conceptual and inter-conceptual connections' (Carrilo-Yanez et al.,2018, p.8); on the other hand, its lack of explicit reference to HCK or ideas of the horizon outside of its literature review prevents an independent coding for any implicit concept of HCK which it could hold. This inhibits any conclusion that could be drawn from the comparison of the two constructs big ideas and HCK. An example of an article rejected for no clear understanding of big ideas is Ball (1993). While there is a singular occurrence of the phrase 'big ideas' in viewing "students as capable of thinking about big and complicated

ideas" (Ball 1993, p. 384), the notion was not explicitly discussed. After obtaining the resulting set of articles (see Tables 1 and 2), we conducted a vertical or within-case analysis followed by a horizontal or cross-case analysis (Miles et al. 2014) for each of the concepts 'big ideas' and 'horizon content knowledge' respectively.

Table 1

Conceptualisations of big ideas in review

Author	Conceptions of "big ideas"		
	Connectivity of knowledge	Disciplinary practices	Setting up powerful teaching moments
Hurst (2015)	See mathematics as 'coherent set of ideas'. Encourage deep understanding of math: enhance transfer, promote memory, reduce amount to be remembered, how topics are connected across years. Knowing 'about' the link rather than knowing a particular link (p. 2).	Problem solving skills and other 'big process ideas' need to be at the heart of teaching and learning. Includes deciding how to tackle problems, gather and organise data, represent and communicate findings. Could be reflected in practices documented in syllabi, e.g. ACARA (p. 9).	
Hurst (2017)	Grants an ability to shift between 'inner' and 'outer' horizons, which respectively denote objects' properties and connections to larger mathematical structures (p. 117).		"It is the 'enabler' that allows teachers to set up, the contingent moments that are the essence of powerful teaching" (p. 117)
Seaman and Szydlik (2007)		Mathematical sophistication: beliefs about nature of mathematical behaviour, values concerning what it means to know mathematics, and particularly in avenues of experiencing mathematics objects and in distinctions about language	
		Explicitly identified some fundamental norms of the community of mathematicians and demonstrated how these norms can help to understand why many preservice teachers find mathematics difficult (p. 170)	
Quebec Fuentes and Ma (2018)		Norms of discussions specific to the field of mathematics, sociomathematical norms, including what makes various explanations mathematically different, sophisticated, efficient and/or acceptable (p. 11)	

Table 2

Conceptualisations of 'horizon content knowledge' in review

Author	Conceptions of *horizon content knowledge*		
	Connected content knowledge based on big ideas	Mathematical knowledge to situated *mathematical horizons* of students in terms of their understanding	Awareness of the affordances of mathematical competencies to highlight mathematical connections
Hurst (2015b)		Teachers with well-developed *horizon content knowledge* (HCK) are able to look both forwards and backwards from a particular point of mathematical understanding and consider how to help a child to develop new knowledge or to see what understanding might be lacking in order to correct a misconception (p. 8).	
Hurst (2017)	Consists of connected content knowledge based on big ideas and also a sensibility about mathematical proficiencies (p. 120).	Teachers need: knowledge of students' mathematical horizons, situated in terms of his/her mathematical understanding (p. 115).	A sensibility about mathematical proficiencies and processes that can be invoked to help children reach their mathematics horizons and move beyond them (p. 120).
Seaman and Szydlik (2007)	The teacher must understand the rich connections among mathematical ideas (p. 168).	Be better able to identify specific mathematics needs to help children in particular situations (p. 168).	
Quebec Fuentes and Ma (2018)		Having connections to mathematical concepts, as presented in various ways, and requiring metacognition. (p. 15)	Developing an understanding of the specific ways of communication and representation centred on developing particular mathematical ideas as well as constituting the disciplinary discourse of mathematics. (p. 11)

Results and Discussion

Three main strands of big ideas and HCK emerged from our analysis. For both big ideas and HCK, two of three strands were respectively grouped around content knowledge and characterisations of mathematical thinking. In the third strand, HCK is explicitly defined in terms of big ideas or vice-versa, and thus could not be coded independently of the other construct.

Conceptualisations of big ideas

Firstly, big ideas are a "coherent set of ideas" (Hurst 2015b, p. 2) which allow for the connectivity of mathematical understanding. This was coded in two of the four studies. Hurst (2015b), quoting from Charles (2005), states that connectivity is important to "encourage a deep understanding of mathematics, enhance transfer, promote memory and reduce the amount to be remembered" (p. 2). Interestingly, big ideas are distinctive in allowing for cognitive improvements such as improved transfer learning and memory performance through a reduction of cognitive load. In consequence, future research could underpin this conception of big ideas based on empirical work. Further, Hurst notes importantly that "[t]here is not necessarily any one particular way in which content ideas can be linked around big ideas", as the big ideas can be linked together in different ways (Hurst 2015b, p. 2). For example, consider the big idea of 'proportionality'. Depending on the lesson objective, students could be led to the "inner horizon" (Hurst 2017, p. 116) to understand why 3/15 is equal to 1/5, or to the "outer horizon" (Hurst 2017, p. 116) to understand why fractions, decimals, percentages (Hurst 2015b, p. 6) are ultimately different representations of the same mathematical object. Ultimately, this first strand of big ideas emphasises the connectivity of mathematical content knowledge, and could be elaborated in specific versions such as in Charles (2005), Ma (2010), and Clarke et al. (2012).

Secondly, big ideas are disciplinary norms and beliefs about the nature of mathematics, which can be shown to affect mathematical understanding. This strand was evident in three of the four studies reviewed. Seaman and Szydlik (2007) most clearly show this through an inability of preservice elementary teachers to re-construct a correct mathematical understanding of the greatest common divisor, even when given mathematical definitions in a teaching resource. Instead, some preservice elementary teachers cling to a procedural approach to mathematics. This differing view on the nature of mathematics prevented them from even attempting to making sense of the relevant definitions. This was a lack of "mathematical sophistication" (Seaman & Szydlik, 2007, p. 169) on the pre-service elementary teachers' part, as Seaman and Szydlik observe amongst other deviations from a non-exhaustive list of nine disciplinary norms. Further empirical work may strengthen this claim to show how disciplinary norms such as problem-solving habits can improve general mathematical performance. Moreover, a closer look at the three studies indicates different understandings of what constitutes mathematics as a discipline. While Seaman and Szydlik (2007) and Quebec Fuentes and Ma (2018) refer explicitly to university mathematicians, Hurst (2015b) does not explicitly address the possibility that the mathematics education community and, what we loosely call the 'university mathematics' community, could have a differing set of norms. Any conception of big ideas based on disciplinary processes must clarify its definition of 'mathematical discipline' before it can be further demonstrated how such big ideas improve mathematical understanding and learning.

Thirdly, big ideas enable better teaching by setting up "contingent moments that are the essence of powerful teaching" (Hurst, 2017, p. 117). Of the four articles, this was explicated

only in Hurst (2017). This conception of big ideas has a clear link to the context of teaching. The resemblance to HCK is not an accident as in his view, "HCK and big ideas are inextricably linked, or even could be considered as one and the same" (Hurst 2017, p. 114). Whether HCK and big ideas are separate constructs need to be further evaluated. This evaluation could happen on the conceptual front as evidenced by further systematic reviews, or on the empirical front by investigating the separability of big ideas and HCK as constructs.

Conceptualisations of Horizon Content Knowledge

First, one concept of HCK is that it is the knowledge required to situate the "mathematical horizons" of student mathematical understanding. This view was found in all four studies. An example of this is provided by Quebec Fuentes and Ma (2018, p. 19), where an open-ended question is posed about a 'yellow square'. Students were to debate if the square is both a polygon and a quadrilateral, and the teacher needs to work with students' definitions of squares and rectangles in order to convince them that a square is a special kind of rectangle. That is, the teacher's content knowledge about mathematical definitions at different curricular levels are required for teachers to look "both forwards and backwards" (Hurst 2015b, p.8) so that the visual understanding of rectangles is connected with an inclusive definition of rectangles. This strand of HCK is thus characterised by open-ended engagement with students' ideas that does not fall into the other categories of Ball et al.'s (2008) categories in SMK or PCK.

A second related concept is that HCK is a sensibility for mathematical horizons, as understood in the previous sense. HCK consists of ways of representing and communicating mathematical ideas that can help children reach beyond their current mathematical horizon. This view was shared by Quebec Fuentes and Ma (2018) and Hurst (2017). Using the same example from Quebec Fuentes and Ma (2018, p. 19), the crux of this conceptualisation of HCK is in the sensitivity of the teacher to student's open-ended answers about squares. The teacher needs to apply "mathematical proficiencies and processes such as reasoning, justifying, hypothesising and problem-solving" (Hurst , 2017, p. 115) to transform student's answers into precise mathematical language, so that the students can understand that "a square is a special kind of rectangle". Like the difference between the first and second concepts of big ideas, the difference between HCK of the first and second kind is in the focus on the teacher's cognitive processes in navigating mathematics, in contrast to the content knowledge invoked for the same purpose. This parallel distinction will be significant in our later conclusion.

The third concept of HCK is that it consists of "connections and links within and between big ideas" (Seamand & Sydzlik, p. 8). This was also found in Hurst (2017), which held a view that HCK and big ideas might be the same (Hurst 2017, p.114). Again, the link between HCK and big ideas ought to be evaluated empirically as well as theoretically. We attempt to evaluate the latter in the next section.

How can teachers teach towards big ideas in the classroom?

Our review of papers discussing both HCK and big ideas simultaneously reveals that the two concepts are related but ultimately distinct. While both concepts are sometimes used to refer to both the content knowledge and the processes underlying it, our review shows that the focus of the two terms are different. For the big ideas of mathematics, the crux of the concept refers to knowledge and mathematical processes that unify the discipline, whereas, for HCK, the corresponding focus is within the context of teaching: HCK is the knowledge consisting of content knowledge and processes required to diagnose a student's current

mathematical horizon and advance it. The difference that characterises big ideas, in contrast to HCK, is an intentional usage of knowledge and practices from mathematics as a discipline. This makes sense given that HCK was originally a component of MKT. Whilst terminological ambiguity could have diluted its meaning (Jakobsen et al., 2014), HCK cannot be dissociated from teaching contexts. In contrast the construct of 'big ideas' runs in the opposite direction with the mathematical discipline influencing teaching.

The distinction of HCK from big ideas helps explicate how teachers can teach towards big ideas in the classroom. We propose that to teach towards big ideas is to embody the epistemic norms of the mathematical community in the classroom. By epistemic norms, we mean habits of the mathematical community that are demonstrably productive towards the generation of mathematical knowledge and the improvement of learning outcomes. This is supported, firstly, by the non-uniqueness of the mathematical connections between content knowledge, as discussed in the first strand of big ideas. That these connections need not be unique suggests greater importance for the habit of deepening one's mathematical content knowledge. Secondly, the second strand of big ideas emphasises strategies of knowing in mathematics that can be meaningfully brought into the classroom. The embodiment of epistemic norms requires the possession of both these strands of big ideas.

We suggest, then, a two-step characterisation for preparing teachers to 'teach towards big ideas'. First, 'teaching towards big ideas' involves an understanding of how mathematicians think. Whether big ideas are conceived as connective content knowledge (Hurst 2015b) or mathematical habits, characterised as "mathematical sophistication" in Seaman and Szydlik (2007), the teacher has to reflect in order to improve her classroom practice. This involves noticing differences from the mathematical discipline in how they conceptualise mathematical connections and in how they think mathematically. Secondly, 'teaching towards big ideas' needs to be evaluated on empirical metrics such as improved classroom practice and/or student learning outcomes. 'Teaching towards big ideas' takes time for its efficacy to be evaluated, and this evaluation could be incorporated into existing frameworks of professional development. The educative curricular approach of Quebec Fuentes and Ma (2018) is but one example of other approaches to professional development, such as lesson study, that can be pursued and investigated.

Finally, a comparison across the four studies suggests that the two distinctions between HCK and big ideas, and between content and process, are valuable for unifying the mathematics education literature. In our review, the studies that surfaced ran across the literature's breadth. The empirical work of Quebec Fuentes and Ma (2018) and Seaman and Szydlik (2007), and the theoretical works of Hurst, form a closed loop, that can benefit from more work that clarifies both constructs in theory and practice.

Conclusion

In conclusion, we suggest that to teach towards big ideas is to emphasise disciplinary ways of thinking that are empirically demonstrable to be fruitful for the learning of mathematics. Our review was limited by three factors. First, the current range of databases could be extended. Second, literature beyond journals should be considered in a more thorough review. We excluded grey literature including books and dissertations for practicality. There is reason to believe that grey literature may be useful for our research question due to its practitioner-oriented focus. Further, given a relatively new focus on big ideas in the literature, new ideas could be expected to be articulated outside of journal publications. Third, whilst teaching towards big ideas may seem to be a new trend in English language mathematics education journals, a systematic review across multiple languages

may reveal greater insights. In German-language literature, for instance, there has long been a tradition of established mathematicians interacting with their mathematics educators along the lines of "fundamental ideas" in mathematics (Vohns, 2016).

Given that mathematics as a discipline needs to be understood both universally as well as contextually, especially in connection to teaching, 'teaching towards big ideas' may benefit from a closer look at the existing interdisciplinary study of mathematical practice. Historical, sociological, and philosophical standpoints can have meaningful contact with the mathematics education literature in creating an empirically-grounded study of successful and diverse mathematical practices (Hamami & Morris, 2020; Kerkhove & Bendegem, 2007). Disciplinary features such as mathematicians' judgements about the elegance of a proof, explanation and understanding, the visualisation of mathematical objects, and the differences between informal and formal proofs are just some of the topics investigated in this burgeoning focus of interdisciplinary inquiry (Hamami & Morris, 2020), of which their contact with 'teaching towards big ideas' is not coincidental. To advance the 'teaching towards big ideas' successfully, it is suggested for further research to integrate knowledge across disciplinary divides, where meaningful.

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